

Trigonometry

Trigonometric Functions

$$T1. \quad \sin^2 x + \cos^2 x = 1$$

$$T2. \quad \tan^2 x + 1 = \sec^2 x$$

$$T3. \quad \cot^2 x + 1 = \csc^2 x$$

$$T4. \quad \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$T5. \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$T6. \quad \tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$T7. \quad \tan\left(\frac{x}{2}\right) = \frac{\sin x}{1 + \cos x}$$

$$T8. \quad \sin(2x) = 2 \sin x \cos x$$

$$T9. \quad \cos(2x) = \cos^2 x - \sin^2 x$$

$$T10. \quad \sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

$$T11. \quad \cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

$$T12. \quad \sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$$

$$T13. \quad \cos x \cos y = \frac{1}{2}(\cos(x - y) + \cos(x + y))$$

$$\text{T14. } \sin x \cos y = \frac{1}{2}(\sin(x-y) + \sin(x+y))$$

$$\text{T15. } c_1 \cos(\omega t) + c_2 \sin(\omega t) = A \sin(\omega t + \phi),$$

$$\text{where } A = \sqrt{c_1^2 + c_2^2}, \quad \phi = 2 \arctan \frac{c_1}{c_2 + A}$$

Hyperbolic Functions

$$\text{T16. } \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\text{T17. } \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\text{T18. } \cosh^2 x - \sinh^2 x = 1$$

$$\text{T19. } \tanh^2 x + \operatorname{sech}^2 x = 1$$

$$\text{T20. } \coth^2 x - \operatorname{csch}^2 x = 1$$

$$\text{T21. } \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\text{T22. } \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\text{T23. } \tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\text{T24. } \sinh(2x) = 2 \sinh x \cosh x$$

$$\text{T25. } \cosh(2x) = \cosh^2 x + \sinh^2 x$$

$$\text{T26. } \sinh x \sinh y = \frac{1}{2}(\cosh(x+y) - \cosh(x-y))$$

T27. $\cosh x \cosh y = \frac{1}{2}(\cosh(x+y) + \cosh(x-y))$

T28. $\sinh x \cosh y = \frac{1}{2}(\sinh(x+y) + \sinh(x-y))$

Power Series

P1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \quad -\infty < x < \infty$

P2. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \quad -\infty < x < \infty$

P3. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots, \quad -\infty < x < \infty$

P4. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

P5. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots, \quad -1 < x < 1$

P6. $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots, \quad -\infty < x < \infty$

P7. $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots, \quad -\infty < x < \infty$

P8. $\tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

P9. $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$

P10. Taylor Series with remainder:

$$f(x) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x-a)^n + R_{N+1}(x), \quad \text{where}$$

$$R_{N+1}(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} (x-a)^{N+1} \quad \text{for some } \xi \text{ between } a \text{ and } x.$$

Derivative formulas

1. **Product rule:** $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
2. **Chain rule:**

$$f(g(x))' = f'(g(x))g'(x)$$

$$\frac{du}{dt} = \frac{du}{dx} \frac{dx}{dt}$$

3. **Quotient rule:** $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
4. $\frac{d}{dx}(e^{cx}) = c \cdot e^{cx}$
5. $\frac{d}{dx} \sin(cx) = c \cdot \cos(cx)$
6. $\frac{d}{dx} \cos(cx) = -c \cdot \sin(cx)$

Algebra formulas

1. **Quadratic formula:** Roots of $ax^2 + bx + c = 0$ are: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
2. $a^2 - b^2 = (a - b)(a + b)$
3. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ and $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
4. $a^n - b^n = (a - b)(a^{n-1} + ba^{n-2} + \dots + b^{n-2}a + b^{n-1})$

Geometry formulas

1. **area of a triangle** of base b , height h : $A = bh/2$
2. **volume of a sphere** of radius r : $V = 4\pi r^3/3$
2. **surface area of a sphere** of radius r : $V = 4\pi r^2$

Table of Integrals

A constant of integration should be added to each formula. The letters a , b , m , and n denote constants; u and v denote functions of an independent variable such as x .

Standard Integrals

$$\text{I1. } \int u^n du = \frac{u^{n+1}}{n+1}, \quad n \neq -1$$

$$\text{I2. } \int \frac{du}{u} = \ln |u|$$

$$\text{I3. } \int e^u du = e^u$$

$$\text{I4. } \int a^u du = \frac{a^u}{\ln a}, \quad a > 0$$

$$\text{I5. } \int \cos u du = \sin u$$

$$\text{I6. } \int \sin u du = -\cos u$$

$$\text{I7. } \int \sec^2 u du = \tan u$$

$$\text{I8. } \int \csc^2 u du = -\cot u$$

$$\text{I9. } \int \sec u \tan u du = \sec u$$

$$\text{I10. } \int \csc u \cot u du = -\csc u$$

$$\text{I11. } \int \tan u du = -\ln |\cos u|$$

I12. $\int \cot u \, du = \ln |\sin u|$

I13. $\int \sec u \, du = \ln |\sec u + \tan u|$

I14. $\int \csc u \, du = \ln |\csc u - \cot u|$

I15. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right)$

I16. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right)$

I17. $\int u \, dv = uv - \int v \, du$

Integrals involving $au + b$

I18. $\int (au + b)^n \, du = \frac{(au + b)^{n+1}}{(n+1)a}, \quad n \neq -1$

I19. $\int \frac{du}{au + b} = \frac{1}{a} \ln |au + b|$

I20. $\int \frac{u \, du}{au + b} = \frac{u}{a} - \frac{b}{a^2} \ln |au + b|$

I21. $\int \frac{u \, du}{(au + b)^2} = \frac{b}{a^2(au + b)} + \frac{1}{a^2} \ln |au + b|$

I22. $\int \frac{du}{u(au + b)} = \frac{1}{b} \ln \left| \frac{u}{au + b} \right|$

I23. $\int u \sqrt{au + b} \, du = \frac{2(3au - 2b)}{15a^2} (au + b)^{3/2}$

I24. $\int \frac{u \, du}{\sqrt{au + b}} = \frac{2(au - 2b)}{3a^2} \sqrt{au + b}$

I25. $\int u^2 \sqrt{au + b} \, du = \frac{2}{105a^3} (8b^2 - 12abu + 15a^2u^2) (au + b)^{3/2}$

I26.
$$\int \frac{u^2 du}{\sqrt{au+b}} = \frac{2}{15a^3} (8b^2 - 4abu + 3a^2u^2) \sqrt{au+b}$$

Integrals involving $u^2 \pm a^2$

I27.
$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right|$$

I28.
$$\int \frac{u du}{u^2 \pm a^2} = {}^{1/2} \ln |u^2 \pm a^2|$$

I29.
$$\int \frac{u^2 du}{u^2 - a^2} = u + \frac{a}{2} \ln \left| \frac{u-a}{u+a} \right|$$

I30.
$$\int \frac{u^2 du}{u^2 + a^2} = u - a \arctan \left(\frac{u}{a} \right)$$

I31.
$$\int \frac{du}{u(u^2 \pm a^2)} = \pm \frac{1}{2a^2} \ln \left| \frac{u^2}{u^2 \pm a^2} \right|$$

Integrals involving $\sqrt{u^2 \pm a^2}$

I32.
$$\int \frac{u du}{\sqrt{u^2 \pm a^2}} = \sqrt{u^2 \pm a^2}$$

I33.
$$\int u \sqrt{u^2 \pm a^2} du = \frac{1}{3} (u^2 \pm a^2)^{3/2}$$

I34.
$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln \left| u + \sqrt{u^2 \pm a^2} \right|$$

I35.
$$\int \frac{u^2 du}{\sqrt{u^2 \pm a^2}} = \frac{u}{2} \sqrt{u^2 \pm a^2} \mp \frac{a^2}{2} \ln \left| u + \sqrt{u^2 \pm a^2} \right|$$

I36.
$$\int \frac{du}{u \sqrt{u^2 + a^2}} = \frac{1}{a} \ln \left| \frac{u}{a + \sqrt{u^2 + a^2}} \right|$$

I37.
$$\int \frac{du}{u \sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \left(\frac{u}{a} \right)$$

$$\text{I38. } \int \frac{du}{u^2\sqrt{u^2 \pm a^2}} = \mp \frac{\sqrt{u^2 \pm a^2}}{a^2 u}$$

$$\text{I39. } \int \sqrt{u^2 \pm a^2} du = \frac{u}{2} \sqrt{u^2 \pm a^2} \pm \frac{a^2}{2} \ln \left| u + \sqrt{u^2 \pm a^2} \right|$$

$$\text{I40. } \int u^2 \sqrt{u^2 \pm a^2} du = \frac{u}{4} (u^2 \pm a^2)^{3/2} \mp \frac{a^2 u}{8} \sqrt{u^2 \pm a^2} - \frac{a^4}{8} \ln \left| u + \sqrt{u^2 \pm a^2} \right|$$

$$\text{I41. } \int \frac{\sqrt{u^2 + a^2}}{u} du = \sqrt{u^2 + a^2} - a \ln \left| \frac{a + \sqrt{u^2 + a^2}}{u} \right|$$

$$\text{I42. } \int \frac{\sqrt{u^2 - a^2}}{u} du = \sqrt{u^2 - a^2} - a \operatorname{arcsec} \left(\frac{u}{a} \right)$$

$$\text{I43. } \int \frac{\sqrt{u^2 \pm a^2}}{u^2} du = -\frac{\sqrt{u^2 \pm a^2}}{u} + \ln \left| u + \sqrt{u^2 \pm a^2} \right|$$

Integrals involving $\sqrt{a^2 - u^2}$

$$\text{I44. } \int \sqrt{a^2 - u^2} du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin \left(\frac{u}{a} \right)$$

$$\text{I45. } \int \frac{u du}{\sqrt{a^2 - u^2}} = -\sqrt{a^2 - u^2}$$

$$\text{I46. } \int u \sqrt{a^2 - u^2} du = -\frac{1}{3} (a^2 - u^2)^{3/2}$$

$$\text{I47. } \int \frac{\sqrt{a^2 - u^2}}{u} du = \sqrt{a^2 - u^2} - a \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right|$$

$$\text{I48. } \int \frac{du}{u \sqrt{a^2 - u^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right|$$

$$\text{I49. } \int u^2 \sqrt{a^2 - u^2} du = -\frac{u}{4} (a^2 - u^2)^{3/2} + \frac{a^2 u}{8} \sqrt{a^2 - u^2} + \frac{a^4}{8} \arcsin \left(\frac{u}{a} \right)$$

$$\text{I50. } \int \frac{\sqrt{a^2 - u^2}}{u^2} du = -\frac{\sqrt{a^2 - u^2}}{u} - \arcsin \left(\frac{u}{a} \right)$$

$$\text{I51. } \int \frac{u^2 du}{\sqrt{a^2 - u^2}} = -\frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin\left(\frac{u}{a}\right)$$

$$\text{I52. } \int \frac{du}{u^2 \sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u}$$

Integrals involving trigonometric functions

$$\text{I53. } \int \sin^2(au) du = \frac{u}{2} - \frac{\sin(2au)}{4a}$$

$$\text{I54. } \int \cos^2(au) du = \frac{u}{2} + \frac{\sin(2au)}{4a}$$

$$\text{I55. } \int \sin^3(au) du = \frac{1}{a} \left(\frac{\cos^3(au)}{3} - \cos(au) \right)$$

$$\text{I56. } \int \cos^3(au) du = \frac{1}{a} \left(\sin(au) - \frac{\sin^3(au)}{3} \right)$$

$$\text{I57. } \int \sin^2(au) \cos^2(au) du = \frac{u}{8} - \frac{1}{32a} \sin(4au)$$

$$\text{I58. } \int \tan^2(au) du = \frac{1}{a} \tan(au) - u$$

$$\text{I59. } \int \cot^2(au) du = -\frac{1}{a} \cot(au) - u$$

$$\text{I60. } \int \sec^3(au) du = \frac{1}{2a} \sec(au) \tan(au) + \frac{1}{2a} \ln |\sec(au) + \tan(au)|$$

$$\text{I61. } \int \csc^3(au) du = -\frac{1}{2a} \csc(au) \cot(au) + \frac{1}{2a} \ln |\csc(au) - \cot(au)|$$

$$\text{I62. } \int u \sin(au) du = \frac{1}{a^2} (\sin(au) - au \cos(au))$$

$$\text{I63. } \int u \cos(au) du = \frac{1}{a^2} (\cos(au) + au \sin(au))$$

$$\text{I64. } \int u^2 \sin(au) du = \frac{1}{a^3} (2au \sin(au) - (a^2 u^2 - 2) \cos(au))$$

$$\text{I65. } \int u^2 \cos(au) du = \frac{1}{a^3} (2au \cos(au) + (a^2 u^2 - 2) \sin(au))$$

$$\text{I66. } \int \sin(au) \sin(bu) du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)}, \quad a^2 \neq b^2$$

$$\text{I67. } \int \cos(au) \cos(bu) du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)}, \quad a^2 \neq b^2$$

$$\text{I68. } \int \sin(au) \cos(bu) du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)}, \quad a^2 \neq b^2$$

$$\text{I69. } \int \sin^n u du = -\frac{1}{n} \sin^{n-1} u \cos u + \frac{n-1}{n} \int \sin^{n-2} u du$$

Integrals involving hyperbolic functions

$$\text{I70. } \int \sinh(au) du = \frac{1}{a} \cosh(au)$$

$$\text{I71. } \int \sinh^2(au) du = \frac{1}{4a} \sinh(2au) - \frac{u}{2}$$

$$\text{I72. } \int \cosh(au) du = \frac{1}{a} \sinh(au)$$

$$\text{I73. } \int \cosh^2(au) du = \frac{u}{2} + \frac{1}{4a} \sinh(2au)$$

$$\text{I74. } \int \sinh(au) \cosh(bu) du = \frac{\cosh((a+b)u)}{2(a+b)} + \frac{\cosh((a-b)u)}{2(a-b)}$$

$$\text{I75. } \int \sinh(au) \cosh(au) du = \frac{1}{4a} \cosh(2au)$$

$$\text{I76. } \int \tanh u du = \ln(\cosh u)$$

$$\text{I77. } \int \operatorname{sech} u du = \arctan(\sinh u) = 2 \arctan(e^u)$$

Integrals involving exponential functions

$$\text{I78. } \int ue^{au} du = \frac{e^{au}}{a^2}(au - 1)$$

$$\text{I79. } \int u^2 e^{au} du = \frac{e^{au}}{a^3} (a^2 u^2 - 2au + 2)$$

$$\text{I80. } \int u^n e^{au} du = \frac{1}{a} u^n e^{au} - \frac{n}{a} \int u^{n-1} e^{au} du$$

$$\text{I81. } \int e^{au} \sin(bu) du = \frac{e^{au}}{a^2 + b^2} (a \sin(bu) - b \cos(bu))$$

$$\text{I82. } \int e^{au} \cos(bu) du = \frac{e^{au}}{a^2 + b^2} (a \cos(bu) + b \sin(bu))$$

Integrals involving inverse trigonometric functions

$$\text{I83. } \int \arcsin\left(\frac{u}{a}\right) du = u \arcsin\left(\frac{u}{a}\right) + \sqrt{a^2 - u^2}$$

$$\text{I84. } \int \arccos\left(\frac{u}{a}\right) du = u \arccos\left(\frac{u}{a}\right) - \sqrt{a^2 - u^2}$$

$$\text{I85. } \int \arctan\left(\frac{u}{a}\right) du = u \arctan\left(\frac{u}{a}\right) - \frac{a}{2} \ln(a^2 + u^2)$$

Integrals involving inverse hyperbolic functions

$$\text{I86. } \int \operatorname{arcsinh}\left(\frac{u}{a}\right) du = u \operatorname{arcsinh}\left(\frac{u}{a}\right) - \sqrt{u^2 + a^2}$$

$$\text{I87. } \begin{aligned} \int \operatorname{arccosh}\left(\frac{u}{a}\right) du &= u \operatorname{arccosh}\left(\frac{u}{a}\right) - \sqrt{u^2 - a^2} & \operatorname{arccosh}\left(\frac{u}{a}\right) > 0; \\ &= u \operatorname{arccosh}\left(\frac{u}{a}\right) + \sqrt{u^2 - a^2} & \operatorname{arccosh}\left(\frac{u}{a}\right) < 0. \end{aligned}$$

$$\text{I88. } \int \operatorname{arctanh}\left(\frac{u}{a}\right) du = u \operatorname{arctanh}\left(\frac{u}{a}\right) + \frac{a}{2} \ln(a^2 - u^2)$$

Integrals involving logarithm functions

I89. $\int \ln u \, du = u(\ln u - 1)$

I90. $\int u^n \ln u \, du = u^{n+1} \left[\frac{\ln u}{n+1} - \frac{1}{(n+1)^2} \right], \quad n \neq -1$

Wallis' Formulas

I91.
$$\begin{aligned} \int_0^{\pi/2} \sin^m x \, dx &= \int_0^{\pi/2} \cos^m x \, dx \\ &= \frac{(m-1)(m-3)\dots(2 \text{ or } 1)}{m(m-2)\dots(3 \text{ or } 2)} k, \end{aligned}$$

where $k = 1$ if m is odd and $k = \pi/2$ if m is even.

I92.
$$\begin{aligned} \int_0^{\pi/2} \sin^m x \cos^n x \, dx &= \\ &\frac{(m-1)(m-3)\dots(2 \text{ or } 1)(n-1)(n-3)\dots(2 \text{ or } 1)}{(m+n)(m+n-2)\dots(2 \text{ or } 1)} k, \end{aligned}$$

where $k = \pi/2$ if both m and n are even and $k = 1$ otherwise.

Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt, \quad x > 0$$

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(n+1) = n!, \quad \text{if } n \text{ is a non-negative integer}$$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^L f(x) dx, \\ a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \\ b_n &= \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx. \end{aligned}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ni\pi x/L}$$

where

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-ni\pi x/L} dx.$$

Bessel Functions

ν = arbitrary real number; n = integer

1. Definition.

$$x^2 y'' + xy' + (\lambda^2 x^2 - \nu^2)y = 0 \text{ has solutions}$$

$$y = y_\nu(\lambda x) = c_1 J_\nu(\lambda x) + c_2 J_{-\nu}(\lambda x), \quad (\nu \neq n),$$

where

$$J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m t^{\nu+2m}}{2^{\nu+2m} m! \Gamma(\nu+m+1)}.$$

2. General Properties.

$$J_{-n}(t) = (-1)^n J_n(t); \quad J_0(0) = 1; \quad J_n(0) = 0, \quad (n \geq 1).$$

3. Identities.

$$(a) \quad \frac{d}{dt}[t^\nu J_\nu(t)] = t^\nu J_{\nu-1}(t).$$

$$(b) \quad \frac{d}{dt}[t^{-\nu} J_\nu(t)] = -t^{-\nu} J_{\nu+1}(t); \quad \frac{d}{dt} J_0(t) = -J_1(t).$$

(c)

$$\begin{aligned} \frac{d}{dt} J_\nu(t) &= {}^{1/2} (J_{\nu-1}(t) - J_{\nu+1}(t)) \\ &= J_{\nu-1}(t) - \frac{\nu}{t} J_\nu(t) \\ &= \frac{\nu}{t} J_\nu(t) - J_{\nu+1}(t). \end{aligned}$$

(d) Recurrence Relation.

$$J_{\nu+1}(t) = \frac{2\nu}{t} J_\nu(t) - J_{\nu-1}(t)$$

4. Orthogonality.

Solutions $y_\nu(\lambda_0 x)$, $y_\nu(\lambda_1 x)$, \dots , $y_\nu(\lambda_n x)$, \dots of the differential system

$$x^2 y'' + xy' + (\lambda^2 x^2 - \nu^2) y = 0, \quad x_1 \leq x \leq x_2$$

$$a_k y_\nu(\lambda x_k) - b_k \left(\frac{d}{dx} y_\nu(\lambda x) \right) \Big|_{x=x_k} = 0, \quad k = 1, 2$$

have the orthogonality property:

$$\int_{x_1}^{x_2} x y_\nu(\lambda_n x) y_\nu(\lambda_m x) dx = 0, \quad (m \neq n)$$

and

$$\begin{aligned} \int_{x_1}^{x_2} x y_\nu^2(\lambda_m x) dx &= \\ \frac{1}{2\lambda_m^2} \left[(\lambda_m^2 x^2 - \nu^2) y_\nu^2(\lambda_m x) + x^2 \left(\frac{d}{dx} y_\nu(\lambda_m x) \right)^2 \right]_{x_1}^{x_2}, & \quad (m = n). \end{aligned}$$

5. Integrals.

$$(a) \quad \int t^\nu J_{\nu-1}(t) dt = t^\nu J_\nu(t) + C$$

$$(b) \quad \int t^{-\nu} J_{\nu+1}(t) dt = -t^{-\nu} J_\nu(t) + C$$

$$(c) \quad \int t J_0(t) dt = t J_1(t) + C$$

$$(d) \quad \int t^3 J_0(t) dt = (t^3 - 4t) J_1(t) + 2t^2 J_0(t) + C$$

$$(e) \quad \int t^2 J_1(t) dt = 2t J_1(t) - t^2 J_0(t) + C$$

$$(f) \quad \int t^4 J_1(t) dt = (4t^3 - 16t) J_1(t) - (t^4 - 8t^2) J_0(t) + C$$

Zeros and Associated Values of Bessel Functions				
α	$j_{0,\alpha}$	$J'_0(j_{0,\alpha})$	$j_{1,\alpha}$	$J'_1(j_{1,\alpha})$
1	2.40483	-0.519147	3.83170	-0.402760
2	5.52008	0.340265	7.01559	0.300116
3	8.65373	-0.271452	10.1735	-0.249705
4	11.7915	0.232460	13.3237	0.218359
5	14.9309	-0.206546	16.4706	-0.196465
6	18.0711	0.187729	19.6159	0.180063
7	21.2116	-0.173266	22.7601	-0.167185
8	24.3525	0.161702	25.9037	0.156725
9	27.4935	-0.152181	29.0468	-0.148011
10	30.6346	0.144166	32.1897	0.140606
11	33.7758	-0.137297	35.3323	-0.134211
12	36.9171	0.131325	38.4748	0.128617
13	40.0584	-0.126069	41.6171	-0.123668
14	43.1998	0.121399	44.7593	0.119250
15	46.3412	-0.117211	47.9015	-0.115274
16	49.4826	0.113429	51.0435	0.111670
17	52.6241	-0.109991	54.1856	-0.108385
18	55.7655	0.106848	57.3275	0.105374
19	58.9070	-0.103960	60.4695	-0.102601
20	62.0485	0.101293	63.6114	0.100035

Table of Laplace Transforms

	$f(t) \equiv \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) \equiv \mathcal{L}\{f(t)\}(s)$
L1.	$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$
L2.	$1, H(t), U(t)$	$\frac{1}{s}$
L3.	$U(t - a)$	$\frac{e^{-as}}{s}$
L4.	$t^n \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$
L5.	$t^a \quad (a > -1)$	$\frac{\Gamma(a + 1)}{s^{a+1}}$
L6.	e^{at}	$\frac{1}{s - a}$
L7.	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
L8.	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
L9.	$f'(t)$	$sF(s) - f(0)$
L10.	$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
L11.	$t^n f(t) \quad (n = 1, 2, 3, \dots)$	$(-1)^n F^{(n)}(s)$
L12.	$e^{at} f(t)$	$F(s - a)$
L13.	$e^{at} \mathcal{L}^{-1}\{F(s + a)\}$	$F(s)$
L14.	$f(t + P) = f(t)$	$\frac{\int_0^P e^{-st} f(t) dt}{1 - e^{-sP}}$

Table of Laplace Transforms

	$f(t) \equiv \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) \equiv \mathcal{L}\{f(t)\}(s)$
L15.	$f(t)U(t-a)$	$e^{-as}\mathcal{L}\{f(t+a)\}$
L16.	$f(t-a)U(t-a)$	$e^{-as}F(s)$
L17.	$\int_0^t f(z) dz$	$\frac{F(s)}{s}$
L18.	$\int_0^t f(z)g(t-z) dz$	$F(s)G(s)$
L19.	$\frac{f(t)}{t}$	$\int_s^\infty F(z) dz$
L20.	$\frac{1}{a} (e^{at} - 1)$	$\frac{1}{s(s-a)}$
L21.	$t^n e^{at}, n = 1, 2, 3, \dots$	$\frac{n!}{(s-a)^{n+1}}$
L22.	$\frac{e^{at} - e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$
L23.	$\frac{ae^{at} - be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$
L24.	$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
L25.	$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$
L26.	$\sin(\omega t) - \omega t \cos(\omega t)$	$\frac{2\omega^3}{(s^2 + \omega^2)^2}$
L27.	$t \sin(\omega t)$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
L28.	$\sin(\omega t) + \omega t \cos(\omega t)$	$\frac{2\omega s^2}{(s^2 + \omega^2)^2}$

Table of Laplace Transforms

	$f(t) \equiv \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) \equiv \mathcal{L}\{f(t)\}(s)$
L29.	$\frac{b \sin(at) - a \sin(bt)}{ab(b^2 - a^2)}$	$\frac{1}{(s^2 + a^2)(s^2 + b^2)}$
L30.	$\frac{\cos(at) - \cos(bt)}{b^2 - a^2}$	$\frac{s}{(s^2 + a^2)(s^2 + b^2)}$
L31.	$\frac{a \sin(at) - b \sin(bt)}{a^2 - b^2}$	$\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}$
L32.	$e^{-bt} \sin(\omega t)$	$\frac{\omega}{(s + b)^2 + \omega^2}$
L33.	$e^{-bt} \cos(\omega t)$	$\frac{s + b}{(s + b)^2 + \omega^2}$
L34.	$1 - \cos(\omega t)$	$\frac{\omega^2}{s(s^2 + \omega^2)}$
L35.	$\omega t - \sin(\omega t)$	$\frac{\omega^3}{s^2(s^2 + \omega^2)}$
L36.	$\delta(t - a)$	$e^{-sa} \quad a > 0, s > 0$
L37.	$\delta(t - a)f(t)$	$f(a)e^{-sa} \quad a > 0, s > 0$

Partial fraction decomposition (PFD) of $N(x)/D(x)$: Let $N(x)$ be a polynomial of lower degree than another polynomial $D(x)$.

1. Factor $D(x)$ into a product of distinct terms of the form $(ax + b)^r$ or $(ax^2 + bx + c)^s$, for some integers $r > 0, s > 0$. For each term $(ax + b)^r$ the PFD of $N(x)/D(x)$ contains a sum of terms of the form $\frac{A_1}{(ax+b)} + \dots + \frac{A_r}{(ax+b)^r}$ for some constants A_i , and for each term $(ax^2 + bx + c)^r$ the PFD of $N(x)/D(x)$ contains a sum of terms of the form $\frac{B_1x+C_1}{(ax^2+bx+c)} + \dots + \frac{B_sx+C_s}{(ax^2+bx+c)^s}$. for some constants B_i, C_j . $\frac{N(x)}{D(x)}$ is the sum of all these.

2. The next step is to solve for these A 's, B 's, C 's occurring in the numerators. Cross multiply both sides by $D(x)$ and expand out the resulting polynomial identity for $N(x)$ in terms of the A 's, B 's, C 's. Equating coefficients of powers of x on both sides gives rise to a linear system of equations for the A 's, B 's, C 's which you can solve. (3-1-2005)